Ermakov approach for empty FRW minisuperspace oscillators

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The formal Ermakov approach for empty FRW minisuperspace cosmological models of arbitrary Hartle-Hawking factor ordering and the corresponding squeezing features are briefly discussed as a possible means of describing cosmological evolution.

The introduction of Ermakov-type (adiabatic) invariants [1] may prove a useful method of investigating evolutionary and chaotic dynamics problems in the "quantum" cosmological framework [2]. Moreover, the method of adiabatic invariants is intimately related to geometrical angles and phases [3], so that one may think of cosmological Hannay's angles as well as various types of topological phases as those of Berry and Pancharatnam [4].

My purpose in the following is to apply the formal Ermakov scheme to the simplest cosmological oscillators, namely the empty Friedmann-Robertson-Walker (FRW) "quantum" universes. The Wheeler-DeWitt (WDW) minisuperspace equation for that case can be written down as follows

$$\frac{d^2\Psi}{d\Omega^2} + Q\frac{d\Psi}{d\Omega} - e^{-4\Omega}\Psi(x) = 0 \ , \tag{1}$$

where Q is the Hartle-Hawking parameter for the factor ordering [5] that is kept as a free parameter, and Ω is Misner's time [6].

Eq. (1) can be mapped in a known way [7] to the canonical equations for a classical point particle of mass $M = e^{Q\Omega}$ and taking $\Psi = q$ as a generalized coordinate and $\Psi^{'}$ as the momentum, leading to

$$\frac{dq}{d\Omega} = e^{-Q\Omega}p \tag{2}$$

$$\frac{dp}{d\Omega} = e^{(Q-4)\Omega}q \ . \tag{3}$$

These equations describe the canonical motion for a classical point universe as derived from the time-dependent Hamiltonian (of the inverted oscillator type [8])

$$H(\Omega) = e^{-Q\Omega}\frac{p^2}{2} - e^{(Q-4)\Omega}\frac{q^2}{2} \ . \tag{4}$$

For this FRW Hamiltonian the triplet of phase-space functions $T_1 = \frac{p^2}{2}$, $T_2 = pq$, and $T_3 = \frac{q^2}{2}$ forms a dynamical Lie algebra (i.e., $H = \sum_n h_n(\Omega)T_n(p,q)$) which is closed with respect to the Poisson bracket, or more exactly $\{T_1, T_2\} = -2T_1$, $\{T_2, T_3\} = -2T_3$, $\{T_1, T_3\} = -T_2$. The FRW Hamiltonian can be written down as $H = e^{-Q\Omega}T_1 - e^{(Q-4)\Omega}T_3$. The Ermakov invariant I

belongs to the dynamical algebra

$$I = \sum_r \epsilon_r(\Omega)T_r \ , \tag{5}$$

and by means of

$$\frac{\partial I}{\partial \Omega} = -\{I, H\} \ , \tag{6}$$

one is led to the following equations for the unknown functions $\epsilon_r(\Omega)$

$$\dot{\epsilon}_r + \sum_n \left[\sum_m C^r_{nm}h_m(\Omega) \right]\epsilon_n = 0 \ , \tag{7}$$

where C^r_{nm} are the structure constants of the Lie algebra that have been already given above. Thus, we get

$$\begin{aligned}
\dot{\epsilon}_1 &= -2e^{-Q\Omega}\epsilon_2 \\
\dot{\epsilon}_2 &= -e^{(Q-4)\Omega}\epsilon_1 - e^{-Q\Omega}\epsilon_3 \\
\dot{\epsilon}_3 &= -2e^{(Q-4)\Omega}\epsilon_2 \ .
\end{aligned} \tag{8}$$

The solution of this system can be readily obtained by setting $\epsilon_1 = \rho^2$ giving $\epsilon_2 = -e^{Q\Omega}\rho\dot{\rho}$ and $\epsilon_3 = e^{2Q\Omega}\dot{\rho}^2 + \frac{1}{\rho^2}$, where ρ is the solution of the Milne-Pinney equation [9]

$$\ddot{\rho} + Q\dot{\rho} - e^{-4\Omega}\rho = \frac{e^{-2Q\Omega}}{\rho^3} \ . \tag{9}$$

In terms of the function $\rho(\Omega)$ the Ermakov invariant can be written as follows [10]

$$I = \frac{(\rho p - e^{Q\Omega}\dot{\rho}q)^2}{2} + \frac{q^2}{2\rho^2} \ . \tag{10}$$

Next, we calculate the time-dependent generating function allowing one to pass to new canonical variables for which I is chosen to be the new "momentum"

$$S(q, I, \vec{\epsilon}(\Omega)) = \int^q dq^{'} p(q^{'}, I, \vec{\epsilon}(\Omega)) \ , \tag{11}$$

leading to

$$S(q, I, \vec{\epsilon}(\Omega)) = e^{Q\Omega} \frac{q^2}{2} \frac{\dot{\rho}}{\rho} + I \arctan\left[\frac{q}{\sqrt{2I\rho^2 - q^2}}\right] +$$
$$+ \frac{q\sqrt{2I\rho^2 - q^2}}{2\rho^2} , \tag{12}$$

where we have put to zero the constant of integration. Thus

$$\theta = \frac{\partial S}{\partial I} = \arctan\left(\frac{q}{\sqrt{2I\rho^2 - q^2}}\right) . \tag{13}$$

Moreover, the canonical variables are now

$$q = \rho\sqrt{2I} \sin\theta , \tag{14}$$

and

$$p = \frac{\sqrt{2I}}{\rho}\left(\cos\theta + e^{Q\Omega}\dot{\rho}\rho\sin\theta\right) . \tag{15}$$

The dynamical angle will be

$$\Delta\theta^d = \int_{\Omega_0}^{\Omega}\langle\frac{\partial H_{\text{new}}}{\partial I}\rangle d\Omega' = \int_0^{\Omega}\left[\frac{e^{-Q\Omega'}}{\rho^2} - \frac{\rho^2}{2}\frac{d}{d\Omega'}\left(\frac{\dot{\rho}}{\rho}\right)\right]d\Omega' , \tag{16}$$

whereas the geometrical angle reads

$$\Delta\theta^g = \frac{1}{2}\int_{\Omega_0}^{\Omega}\left[\frac{d}{d\Omega'}(e^{Q\Omega'}\dot{\rho}\rho) - 2e^{Q\Omega'}\dot{\rho}^2\right]d\Omega' . \tag{17}$$

The total change of angle will be

$$\Delta\theta = \int_{\Omega_0}^{\Omega}\frac{e^{-Q\Omega'}}{\rho^2}d\Omega' . \tag{18}$$

On the Misner time axis, going to $-\infty$ means going to the origin of the universe, whereas $\Omega_0 = 0$ means the present epoch. Thus, using these cosmological limits in Eq. (18) one can see that the total change of angle can be written as the Laplace transform (up to a sign) of the inverse square of the Milne-Pinney function, $\Delta\theta = -L_{1/\rho^2}(Q)$.

Passing now to the quantum Ermakov problem we turn q and p into quantum-mechanical operators, $\hat{q}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial q}$, but keeping the auxiliary function ρ as a c number. The Ermakov invariant is a constant Hermitian operator (if one works with real Milne-Pinney functions)

$$\frac{dI}{d\Omega} = \frac{\partial I}{\partial\Omega} + \frac{1}{i\hbar}[I, H] = 0 \tag{19}$$

and the FRW time-dependent Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial\Omega}|\psi(q,\Omega)\rangle = H(\Omega)|\psi(q,\Omega)\rangle . \tag{20}$$

The goal now is to find the eigenvalues of I

$$I|\phi_n(q,\Omega)\rangle = \kappa_n|\phi_n(q,\Omega)\rangle \tag{21}$$

and to write the explicit superposition form of the general solution of Eq. (20) [11]

$$\psi(q,\Omega) = \sum_n C_n e^{i\alpha_n(\Omega)}\phi_n(q,\Omega) , \tag{22}$$

where C_n are superposition constants, ϕ_n are the (orthonormal) eigenfunctions of I, and the phases $\alpha_n(\Omega)$, (so-called Lewis phases) are to be found from the equation

$$\hbar\frac{d\alpha_n(\Omega)}{d\Omega} = \langle\psi_n|i\hbar\frac{\partial}{\partial\Omega} - H|\psi_n\rangle . \tag{23}$$

The key point for the quantum Ermakov problem is to perform a clever unitary transformation in order to obtain a transformed eigenvalue problem for the new Ermakov invariant $I' = UIU^\dagger$ possessing time-independent eigenvalues. It is easy to get the required unitary transformation as $U = \exp[-\frac{i}{\hbar}(e^{Q\Omega})\frac{\dot{\rho}}{\rho}\frac{\hat{q}^2}{2}]$ and the new Ermakov invariant will be $I' = \frac{\rho^2\hat{p}^2}{2} + \frac{\hat{q}^2}{2\rho^2}$. Therefore, its eigenfunctions are $\propto e^{-\frac{\theta^2}{2\hbar}}H_n(\theta/\sqrt{\hbar})$, where H_n are Hermite polynomials, $\theta = \frac{q}{\rho}$, and the eigenvalues are $\kappa_n = \hbar(n + \frac{1}{2})$. Thus, one can write the eigenfunctions ψ_n as follows

$$\psi_n \propto \frac{1}{\rho^{\frac{1}{2}}}\exp\left(\frac{1}{2}\frac{i}{\hbar}(e^{Q\Omega})\frac{\dot{\rho}}{\rho}q^2\right)\exp\left(-\frac{q^2}{2\hbar\rho^2}\right)H_n\left(\frac{1}{\sqrt{\hbar}}\frac{q}{\rho}\right) . \tag{24}$$

The factor $1/\rho^{1/2}$ has been introduced for normalization reasons. Using these functions and performing simple calculations one is lead to the geometrical phase

$$\alpha_n^g = -\frac{1}{2}\left(n + \frac{1}{2}\right)\int_{\Omega_0}^{\Omega}\left[(\ddot{\rho}\rho) - \dot{\rho}^2\right]d\Omega' . \tag{25}$$

For the general phases entering the Ermakov superposition in Eq. (22) one gets the following result

$$\alpha_n(\Omega) = -(n + \frac{1}{2})\int_{\Omega_0}^{\Omega}\frac{e^{-Q\Omega'}d\Omega'}{\rho^2} \tag{26}$$

and in the cosmological limits one finds once again a Laplace transform of the inverse square of the Milne-Pinney function, $\alpha_n = (n + \frac{1}{2})L_{1/\rho^2}(Q)$.

The Ermakov procedure allows to construct cosmological squeezed states in a very convenient way [12,13]. For this one makes use of the factorization of the Ermakov invariant $I = \hbar(bb^\dagger + \frac{1}{2})$, where

$$b = (2\hbar\rho)^{-1/2}[\rho^{-1/2}q + i\rho^{1/2}(\rho p - e^{Q\Omega}\dot{\rho}q)] \tag{27}$$
$$b^\dagger = (2\hbar\rho)^{-1/2}[\rho^{-1/2}q - i\rho^{1/2}(\rho p - e^{Q\Omega}\dot{\rho}q)] . \tag{28}$$

Let us now consider a reference Misner-time-independent oscillator with the Misner frequency fixed at an arbitrary

epoch Ω_0 for which one can write the common factoring operators

$$a = (2\hbar\omega_0)^{-1/2}[\omega_0 q + ip] \qquad (29)$$

$$a^\dagger = (2\hbar\omega_0)^{-1/2}[\omega_0 q - ip] \; . \qquad (30)$$

The connection between the a and b pairs is given by

$$b(\Omega) = \mu(\Omega)a + \nu(\Omega)a^\dagger \qquad (31)$$

$$b^\dagger(\Omega) = \mu^*(\Omega)a^\dagger + \nu^*(\Omega)a^\dagger \; , \qquad (32)$$

where

$$\mu(\Omega) = (4\omega_0)^{-1/2}[\rho^{-1} - ie^{Q\Omega}\dot\rho + \omega_0\rho] \qquad (33)$$

and

$$\nu(\Omega) = (4\omega_0)^{-1/2}[\rho^{-1} - ie^{Q\Omega}\dot\rho - \omega_0\rho] \qquad (34)$$

fulfill the well-known relationship $|\mu(\Omega)|^2 - |\nu(\Omega)|^2 = 1$. The corresponding uncertainties are known to be $(\Delta q)^2 = \frac{\hbar}{2\omega_0}|\mu - \nu|^2$, $(\Delta p)^2 = \frac{\hbar\omega_0}{2}|\mu + \nu|^2$, and $(\Delta q)(\Delta p) = \frac{\hbar}{2}|\mu + \nu||\mu - \nu|$ showing that in general the Ermakov squeezed states are not minimum uncertainty states [13].

In conclusion, a simple cosmological application of the (classical and quantum) Ermakov procedure has been presented on the base of a classical point particle representation of the FRW WDW equation. It is also to be noted that the Ermakov invariant is equivalent to the Courant-Snyder one [14], allowing thus in a certain sense a beam physics approach to cosmological evolution.

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